Mapping and Multiple-Representations of Contextual Problems As Alternative Ways of Learning About Functions

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Introduction

Difficulties encountered by students in learning mathematics have been the focus of many research efforts throughout the years. This is often the theme of presentations on mathematics learning at conferences, and publications in educational journals that focus on the teaching and learning of mathematics. This theme was also echoed a number of times in the recent VIII Pacific Science Inter-Congress held at the University of the South Pacific from July 13 - 19, 1997 in which a number of presentations in mathematics education ranged from descriptions of the discrepancies that exist between stated objectives for school mathematics and its assessment procedures (Bastik 1997) to the description of the mismatch between students' cultural perspectives and the necessary skills required for understanding abstract mathematics as traditionally taught in schools, (Bakalevu 1997). Other papers described the problems and difficulties encountered by university students while attempting to learn mathematics, (Muralidhar 1997).

In this paper, instead of again describing the scenario of students finding it difficult to make sense of mathematics, the situation is accepted, but it moves beyond that to propose some alternative ways in which students' learning of mathematics can be enhanced and made more meaningful. Firstly, students' understanding of mathematics concepts can be improved by actively situating one's knowledge of a piece of mathematics within a network of conceptual interconnections that can ultimately expand to reflect the conceptual structure of mathematics within a unit. Secondly, students' understanding of mathematics can be expanded by persistently looking for multiple ways of representing given information to gain a more holistic and complete picture of the situation.

The first alternative requires the use of concept maps to map out the

structure of mathematics and Vee maps to guide the analysis of a problem. The second alternative involves the utilization of graphical, numerical, tabular and algebraic representations simultaneously to capture as complete a view as possible of the given information in the contextual problem. The topic "Functions" is chosen as the mathematics content through which these alternative ways will be illustrated. Functions as a topic is selected, since an understanding of the function concept is essential as a foundation for further learning in mathematics.

This paper first summarizes the results obtained from two different studies conducted by the author, followed by a discussion of how the two approaches can be used jointly to provide a powerful and viable way for students to learn mathematics. The first study was conducted to examine ways in which the use of Gowin's Vee and concept maps can influence the teaching and learning of mathematics (Afamasaga-Fuata'i 1997; Fuata'i 1985); and the second study investigated students' strategies in solving contextual problems on quadratic functions, (Afamasaga-Fuata'i 1995, 1992). Each of these studies will be discussed below, separately, followed by a synthesis of how the results can guide teachers' curricula and pedagogical decisions in making classroom mathematics more operational and functional for students.

Concept Mapping and Vee Mapping in Mathematics

The use of concept maps and Vee maps in science and mathematics classrooms as learning aids for students and curricular aids for teachers has been investigated before at both secondary and university level (Cardamone 1975; Chen 1980; Buchweitz 1981; Gurley 1982; Minemier 1983). Cardamone (1975) used the five telling questions (see Fuata'i 1985 and Gowin 1981 for a full description of Vee maps and the five telling questions) to analyze the structure of knowledge in a college level basic mathematics course. Knowing the historical development of a particular mathematics unit facilitated the arrangement of the hierarchical levels and interrelationships of mathematical concepts in his single concept map of a mathematics unit. Cardamone found concept

maps useful as curriculum development and instructional tools, especially in determining the sequence of topics for instruction and as a display tool for the curriculum.

The two studies by Chen (1980) and Buchweitz (1981) showed that Vee mapping can be a powerful tool for curriculum improvement as demonstrated by enhanced student achievement. Both also showed how a systematic analysis of the laboratory guides can identify possible sources of 'misunderstanding' and 'misconception' and subsequently how an improved version can remedy these defects.

Gurley's study (1982), using two comparison groups and two experimental groups, was the first one to investigate the use of both Vee maps and concept maps by students taking a high school biology course for a school year. The results show that Vee and concept mapping students were "better able to demonstrate for themselves the interrelationships between theory and method, ideas and observations. The Vee-instructed students have a better idea of how the laboratory activity supports and relates to chapter information" (Gurley 1982). In addition, students found the learning strategies require "understanding of the subject matter and are hard work" and that they "recognize and value understanding over rote learning".

Minemier (1983), looking at the usefulness of concept mapping in a college level basic skills mathematics course, found that students "initially disliked it [concept mapping], but gradually accepted and saw some value in it. Students can learn to make concept maps of mathematics; and success in mapping comes through understanding of the material and practice".

The above studies clearly show that the use of the mapping strategies can be a viable alternative for students having difficulties in relating mathematics concepts to the algorithms of mathematics. Subsequently, Fuata'i (1985) conducted an empirical study to determine ways in which Vee maps and concept maps can contribute to making mathematics learning more meaningful than routine applications of formulas to problems, and memorization of rules and procedures. The

study was guided by Gowin's educating and Ausubel's assimilation theories, (see Fuatai (1985) for a full discussion of the theories as they relate to the study, or Afamasaga-Fuata'i (1997) for a brief summary). The study was conducted for one school term of fourteen weeks with two groups of form five students attending Samoa College. Analyses of the mathematics curriculum using the Vee map and its telling questions formed the bases of the teacher's class presentations and instructional plans. The treatment group constructed concept maps to illustrate the conceptual structure of mathematics units and to map the interrelationships of concepts in a given problem; they also used Vee maps to analyze problems; and used both maps jointly to determine solutions. In contrast, the comparison group was given traditional instruction; and depended on textbook, class-worked examples, and their own notes for solving problems. Both groups were given the same tests, the same or equivalent assignments, and the same examinations.

A questionnaire was administered twice, before and after the study, to determine students' thinking, feeling, and actions towards mathematics. All students were interviewed at the end of the study. Statistical analyses of the scores showed significant differences between the two groups in all four assignments and one of the three tests. Appendix 1 shows a concept map on linear functions. Appendices 2 and 3 show two methods of solving a problem. The main difference between the two methods is in the selection of appropriate principles to be placed on the left conceptual side of the Vee.

Results from the study showed that students who used the Vee maps and concept maps felt better about mathematics in general; the strategies made mathematics learning 'more interesting' and 'more meaningful'. Students found the concept maps to be more practical in revising, and more effective in exhibiting the conceptual structure of mathematics, than notes; a concise and explicit way of showing the inter-connections between concepts and formulas in a mathematics problem, and more effective in guiding the generation of solutions to novel problems, particularly when used jointly with Vee maps. Furthermore, students found the Vee maps to be much easier than concept maps to draw; effective tools in guiding the critical thinking

process required in analyzing and solving problems; more efficient in exhibiting the conceptual side and methodological side of a problem; and they provided a systematic approach for the analysis of the structure of knowledge in a mathematics unit and problem. Mapping facilitated 'self-educating' and 'self-reliance'. Requiring students to draw maps in assignments forced them to critically analyze the interrelationships of concepts, resulting in improved understanding of mathematics. In contrast, the non-mapping students were easily frustrated if they could not solve novel problems, constantly asked for more notes and more explanations, and continuously required guidance from the teacher. The results of the study implied that Vee and concept mapping is a viable and practical alternative way of learning mathematics, particularly in explicating its conceptual structure and how this conceptual understanding can lead to the generation of solutions to problems.

Students' Strategies for Solving Contextual Problems

The author conducted an in-depth study to investigate students' actions and strategies in solving contextual problems modeled by quadratic functions. Contextual problems with multiple settings were used as critical sites for students' mathematizing activities and a means of: (i) encouraging students' ways of knowing and problem solving; and (ii) giving students a feel for the "smooth change or variation between co-varying quantities" (Afamasaga-Fuata'i 1994; 1992). A multi-representational computer software, Function Probe, including a graph, table and calculator, was used by students. The theoretical framework was Piagetian constructivism, claiming that humans construct knowledge to organize and make sense of their learning. The researcher traced the dialectic between students and their problematics as students 'think aloud'; and modeled students' developmental conceptions of quadratic relationships when solving maximum/minimum problems involving quadratics. A Function Probe tutorial, contextual problems, pre- and post-tests were used to collect data. Four students were selected on the basis of high pre-test scores on linear functions. The methodology was using teaching interviews between a researcher and student. Problem solving sessions were two hours per session, twice a

week for eight weeks. All sessions were audio- and video-taped for analysis.

An example of a problem given to students reads as follows:

> Farmer Joe has records showing that if 25 avocado trees are planted, then each tree yields 500 avocados (on average). For each additional tree planted, the yield decreases by 10 avocados per tree. Determine the number of trees that would maximize total yield.

The four students in the study each interpreted the problem differently but the commonality was that no one initially suggested the usual product form of quadratic functions describing the relationship between total yield and the number of trees. Multiple interpretations of the problem based on their own interpretations of the problem context led students to totally different sets of representations in terms of tables of numerical values generated, and actual procedural operations used. They all started off by numerically estimating the yield as each tree was added and then building up the numerical values until a procedural pattern became clear. Students then attempted to represent these patterns using algebraic equations. Since students were obtaining total yields iteratively, they struggled with the problem of representing these iterations with a formula. One of the students (Mary) overcame this problem by reconsidering an alternative interpretation that was more consistent with the given context, which eventually led her to view total yield as a product of total number of trees times average yield (Afamasaga-Fuata'i 1991,1995). Nan, on the other hand, tried to use first differences between her consecutive total yields to predict when yield would be maximized. However, because she was also carrying over a numerical error from her initial calculations, she continued to pursue the significance of differences much further than the others until she finally developed an equation that predicted first differences between consecutive total yields for any number of trees. Nan used these first differences to estimate where maximizing number of trees should occur. From the resulting pattern, she predicted that yield will maximize when the first difference is zero but was uncertain as to what

this meant in terms of number of trees. Bob, the third student, in contrast, predicted the maximizing number of trees to occur when the rates of change between trees and average yield reflected the rate of decrease of average yield. (See Afamasaga-Fuata'i 1991.)

Data from the three case analyses showed students' conceptualizations of mathematics concepts in realistic situations represented legitimate and viable alternatives to formal views traditionally taught in school mathematics. In contrast to solving maximum problems as product of two linear functions, these students preferred to interpret quadratics as (i) a sum of a constant and two summations; (ii) a sum of a constant, linear function, and simple quadratic function; and (iii) a difference of two summations. By the end of the study, students were characterizing quadratic relationships in terms of: (1) rates of change, (2) symmetry and (3) dimensionality. In rates of change, quadratics had linear first differences (Δy) in contrast to constant Δy with linear relationships; constant second differences $\Delta(\Delta y)$; and product maximized when dimensions reached values in ratio of their rates of changes. Students often used iterative approaches to describe the co-variation between two quantities; typically one increased and the other decreased. Symmetric relations are seen in the distribution of y values in tables and graphs which led to construction of maximum point schemes. The dimensional factor of two was evidenced by (i) two linear variations; (ii) the appearance of 2 in relationships between rates of change and quadratic terms; and (iii) the term x^2.

Ongoing Research

The results of the two studies Fuata'i (1985) and Afamasaga-Fuata'i (1991) suggest that solving contextual problems in conjunction with concept maps and Vee maps to illustrate and exhibit students' understanding of the mathematics that emerge as a result of successfully solving the problems can be a powerful combination. Students may be given contextual problems, first to solve as a means of introducing new concepts, after which they can then be required to use concept maps and Vee maps to illustrate the structure of the mathematics that emerges as a result of solving the problems. Further

use of appropriately selected contextual problems can reinforce and refine students' evolving understanding of a topic.

Ongoing research involves working with small groups of undergraduate students as part of an undergraduate Mathematics semester-long course. This preliminary work at the undergraduate level is two-fold. First, it is done to determine how students at this level solve contextual problems modeled by different types of functional relationships. Secondly, it is done to determine how students use the mapping strategies in analyzing the structure of mathematics for selected topics. The two parts are done separately by the same students but it is their choice to decide whether to use the mapping strategies to exhibit the patterns identified for the first part or not.

Students are given contextual problems modeled by linear functions and then quadratic functions to solve individually as a means of initiating them into the type of problem-solving approach required for the course. Discussion of each student's approach and method of solving the problem is done in a group session only after each student has had a chance to attempt the problems individually. Each student is required to submit their own problem-solving efforts in written form to the researcher. In addition, each student has to justify and defend their own strategies during the group session that follows. Further variations of original conditions in the contextual problem are then assigned for further work as a means of challenging students' tentative conjectures until those they have proposed are consistent with and work for all types of variations. In the next phase of problem solving, students are required to focus on numerical patterns with the rates of changes (first and second differences) to determine ways of predicting the original equations for y in terms of these differences. Group discussions always follow individual efforts until the students have all agreed on what constitutes the best way of representing their conclusions. Consensus is reached and conjectures are accepted as correct if and only if they hold for all types of variations for each particular type of function. The final project for this course requires students to investigate the case of cubic functions in terms of patterns with quadratics and linear functions or any other pattern. This is done as an individual project to be handed

in at the end of the semester. This initial work has shown that students still have difficulties articulating their findings in general terms relative to the class of cubic functions.

In a separate course, these same students had learnt the use of concept maps for illustrating conceptual interrelationships in selected mathematics units and had used both Vee maps and concept maps to solve problems. Topics students have mapped include functions, vectors and matrices, calculus, linear algebra and statistics.

Data collected from both these courses are currently being analyzed to determine the best way of incorporating the mapping strategies as part of students' work in solving contextual problems. Results from these studies are currently being analyzed and will be made available as soon as they are completed.

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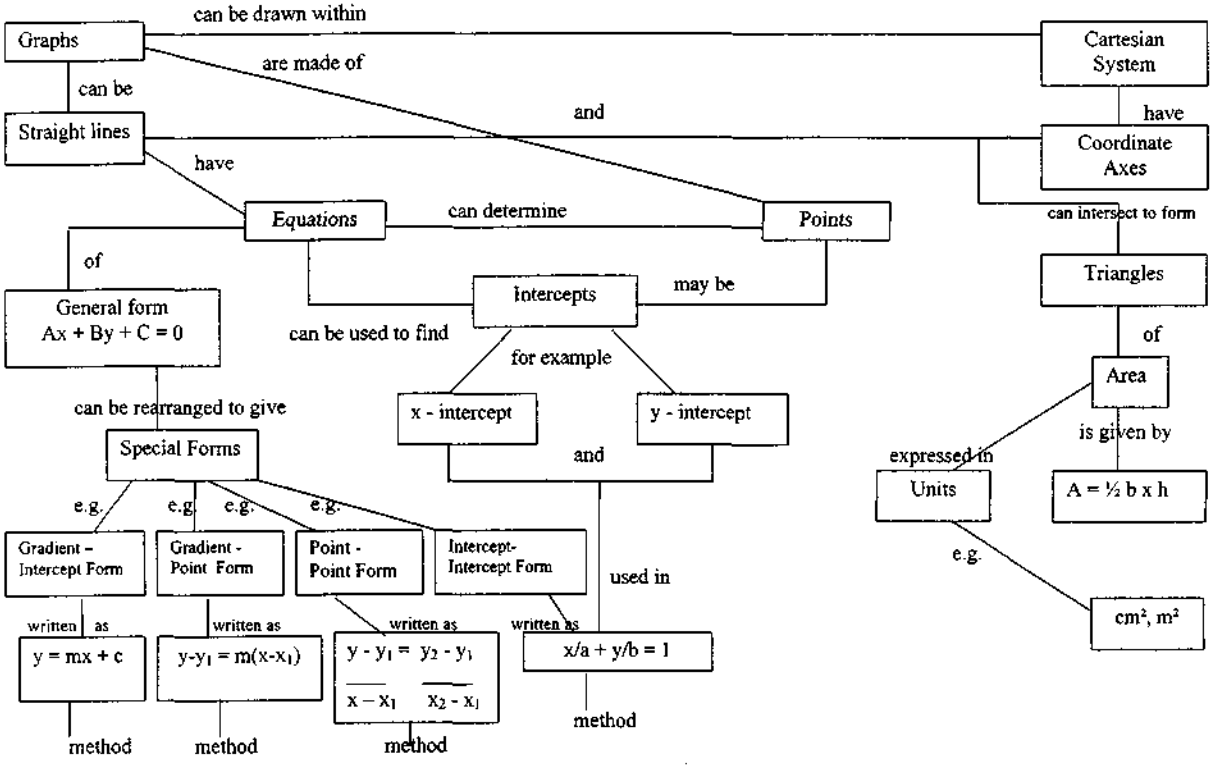
APPENDX 1 : CONCEPT MAP ON LINEAR FUNCTIONS



APPENDIX 2: VEE MAP OF PROBLEMS, METHOD 1

FOCUS QUESTION

What are the equations of the lines passing through *(3, -3)* in general form?

THEORIES

Set theory
--- relations & functions
--- coordinate geometry

PRINCIPLES

1. The general form of equations of a straight line is:
   $Ax + By + C=0$

2. The intercept form of equations of straight lines is: $x/a + y/b = 1$ where $a = x$ - intercept and $b = y$ - intercept

3. Area of triangle is:
   ½ base x height

CONCEPTS

equation, coordinate axes,
lines, point, area,
triangle, base, height, square units

KNOWLEDGE CLAIM

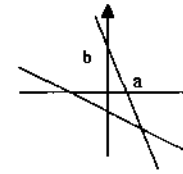
Lines passing through *(3, - 3)* and making triangles of area 6 sq. units with the coordinate axes have equations:
$x + 3y + 6 = 0$ &
$3x + y - 6 = 0$

TRANSFORMATIONS

Area of triangle ½ base x height
$= 1/2ab = 6$ *sq. units;*
$b = 12/a$ ... *(i)*
Using Principle 2; equation of the line is $x/a + y/b = 1$ ....*(ii)*
Substituting (i) in (ii) gives:
$x/a + ay/12 = 1$ ... (iii)
(3; -3) therefore substituting for the values of x and y in (iii) should give:
$3/a - 3a/12 = 1$ ... (iv)
Rearranging and simplifying (iv) gives $a = -6$ and $a = 2$.
Substituting the values of a in (i) and (ii) gives the equations of the lines:

$x + 3y + 6 = 0$ *and*
$3x + y - 6 = 0$

RECORDS



Area = 6 square units

EVENT / OBJECT

Find the equations of the lines which pass through (3, -3) and form with the coordinate axes a triangle of area 6 square units.

APPENDIX 3: VEE MAP OF PROBLEMS, METHOD 2

FOCUS QUESTION

What are the equations of the lines passing through (3, -3) in general form?

THEORIES

Set theory, Number theory
--- relations & functions
--- coordinate geometry

PRINCIPLES

1. The general form of equations of a straight line is $Ax + By + C = 0$
2. The point-point form equations of straight lines is;

$$\frac{(y-y_1)}{(x-x_1)} = \frac{(y_2-y_1)}{(x_2-x_1)}$$

where (x_1, y_1) and (x_2, y_2) are points on the line.

3. Area of triangle is:
   ½ base x height

CONCEPTS

equations, coordinate axes, point, area, triangle, base, height, square units

KNOWLEDGE CLAIM

Lines passing through $(3, -3)$ and making triangles of area 6 sq. units with the coordinate axes have equations:

$3x + y - 6 = 0$ &

$x + 3y + 6 = 0$

TRANSFORMATIONS

From the diagram, a and b are the intercepts, giving the points $(a, 0)$ & $(0,b)$ Using Principle 2, where $x_1 = a$, $y_1 = 0$, $x_2 = 0$, $y_2 = b$

equation of line is:

$$\frac{(y-0)}{(x-a)} = \frac{(b-0)}{(0-a)}$$

which simplifies to:

$y/(x-a) = -b/a$ ... (i)

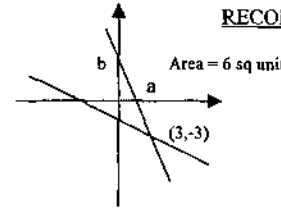
Using Principle 3, $b=12/a$

Substituting b in (i) gives:

$y/(x-a) = -12/a^2$ .... (ii)

Substituting the values of x and y from point (3, -3) in (ii) will give the same values of a as in Method 1 above and consequently the same answers.

RECORDS



EVENT / OBJECT

Find the equations of the lines which pass through (3, -3) and form with the coordinate axes a triangle of area 6 square units.